

The curve complex has dead ends

Joan S. Birman and William W. Menasco

October 26, 2012

Abstract

Let X, Y be vertices in the curve complex $\mathcal{C}(\Sigma)$ of a non-exceptional surface. The vertex Y is a *dead end* with respect to X if there is no extension of geodesics joining X and Y , past Y , to longer geodesics. It has a *double dead end* if it has dead ends at both X and Y . In this paper we will show what has apparently been missed: that there are myriad examples of dead ends and double dead ends in $\mathcal{C}(\Sigma)$, the simplest example being one in which $d(X, Y) = 3$ on a surface of genus 2. A complete picture emerges when we find *nasc* for Y to be a dead end with respect to X and prove that every dead end in $\mathcal{C}(\Sigma)$ has depth 1.

Mathematical Subject Classification 57M99, 20F65

Key words Curve complex, curves on surfaces, dead end of a geodesic, depth of a dead end.

1 Introduction and statement of results

Let $\Sigma = \Sigma_g$ be a closed orientable surface¹ of genus $g \geq 2$. The *curve complex* $\mathcal{C}(\Sigma)$ is a simplicial complex whose vertices are homotopy classes of simple closed curves on Σ_g . (By an abuse of notation, we shall not distinguish between a vertex in $\mathcal{C}(\Sigma)$ and a tight closed curve on Σ representing it.) A collection of $k + 1$ vertices $\{Y_i\}_0^k$ form a k -simplex when their homotopy classes have pairwise-disjoint representatives on Σ . Assign length 1 to every edge in the 1-skeleton of $\mathcal{C}(\Sigma)$. The *distance* $d(X, Y)$ between two vertices is the length of a shortest path in the 1-skeleton of $\mathcal{C}(\Sigma)$ from V to W . The distance function is subtle and interesting. Two types of local pathology in $\mathcal{C}(\Sigma)$ are well-known and at the heart of difficulties in studying $\mathcal{C}(\Sigma)$. The first is that $\mathcal{C}(\Sigma)$ is locally infinite, that is there are infinitely many vertices that are distance 1 from any

¹We make this assumption for brevity. It can surely be relaxed.

given vertex. The second is more subtle: typically, there are infinitely many topologically inequivalent geodesics joining distinct vertices X, Y .

The main result in this paper is that there is additional local pathology that appears to have been missed. A vertex Y is a *dead end* with respect to a vertex X if there is a geodesic \mathcal{G} joining X to Y that cannot be extended past Y . This depends on the choices of X and Y , but not on the choice of \mathcal{G} , for if \mathcal{G} can be extended one vertex past Y to a new vertex Z , with $d(X, Z) = 1 + d(X, Y)$, then adding Z to any geodesic from X to Y will also give a longer geodesic. It has a *double dead end* if it has dead ends at both X and Y . In this paper we will prove that dead ends and double dead ends exist and are even ubiquitous in $\mathcal{C}(\Sigma)$.²

The *depth* of a dead-end in a geodesic of length n is the number of edges one must follow backwards from the dead end to reach a vertex from which the shortened geodesic can be extended to a geodesic of length $\geq n + 1$. When we told several experts about our discovery that $\mathcal{C}(\Sigma)$ has dead ends, the immediate reaction was surprise, followed by a question: ‘What’s the depth?’ We answer that question by giving a universal construction that obtains all pairs (X, Y) with Y a dead end, and using it to prove that every dead end has depth 1.

The basic example: Our initial discovery was a geodesic of length 3 on a surface of genus 2 that turned out to have a dead end. We found it by accident, when we were trying to find examples of geodesics of length 4 by extending ones of length 3, and discovered that in one particular case that was impossible: The left sketch in Figure 1 shows a surface of genus 2, represented as a torus with a disc removed, to which a handle has been attached along the two green curves. The numbers ‘1,2,3’ tell you how to attach it. Capping the boundary with a disc, we obtain the desired closed surface Σ . Simple closed curves X and Y on Σ are illustrated, with Y dividing Σ into two tori. The curves X and Y *fill* Σ_2 , that is every region in the complement of $X \cup Y$ is a disc.³ If two curves fill a surface the distance between them on the curve complex is at least 3 (see [3]). Therefore, to prove that $d(X, Y) = 3$ we need only find a path of length 3 in $\mathcal{C}(\Sigma)$ that joins them.

The arc that’s labeled R is assumed to have been completed to a simple closed curve by another arc on the disc we used to cap the surface. Observe that $Y \cap R = \emptyset$, so that $d(R, Y) = 1$. Checking, we find that $X \cup R$ does not fill Σ . Let $\mathcal{N} = \mathcal{N}(X \cup R)$ be a neighborhood on Σ of $X \cup R$. Then there must be a component of $(\Sigma \setminus \mathcal{N})$ that is not a disc. Choose X_R to be the boundary of this component. Since $X \cap X_R = R \cap X_R = \emptyset$, we have constructed a path

²The apparent conflict between our work and that in [4] comes from the use of the term ‘dead end’ in [4] without definition, resulting in an incorrect description of correct work. Indeed, we will use a key lemma from [4] in this paper.

³Equivalently, A and B fill Σ if the genus of a neighborhood \mathcal{N} in Σ of $A \cup B$ is $g(\Sigma)$. An Euler characteristic count gives the formula $2g = 2 + k - q$, where k is the geometric intersection number of A and B and q is the number of components in $\partial\mathcal{N}$. This gives a constructive way to test whether explicit pairs of curves fill or do not fill Σ .

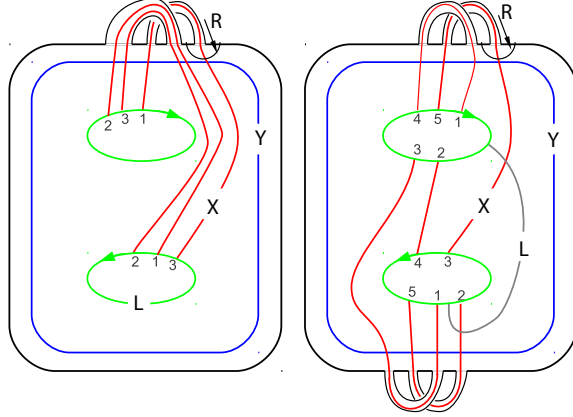


Figure 1: The basic example for $g=2$ and 3

$\mathcal{R} = (X, X_R, R, Y)$ of length 3 and proved that it's a geodesic.

Next, observe that there is a second such length 3 path, because we could equally well have chosen L instead of R and argued as before that $X \cup L$ does not fill, so that we may choose X_L to obtain a geodesic $\mathcal{L} = (X, X_L, L, Y)$. By construction, R and L are separated by Y .

Suppose that the geodesic \mathcal{R} could be extended through Y to Z , where $d(X, Z) = 4$. Then $Z \cap Y = \emptyset$, and since Y separates Σ into two tori $\Sigma^{(1)}$ and $\Sigma^{(2)}$, it follows that Z must be in $\Sigma^{(1)}$ or $\Sigma^{(2)}$. But if it's in $\Sigma^{(1)}$ (resp. $\Sigma^{(2)}$) then (X, X_L, L, Z) (resp. (X, X_R, R, Z)) is a path of length 3 from X to Z . Either way we have shown that Z cannot be more than distance 3 from X . But then no extension exists, and Y is a dead end.

It is clear from the discussion relating to our very simple example, that the following holds:

Lemma 1 *Sufficient conditions for Y to be a dead end with respect to X are: (a) The curve Y separates Σ , and (b) There exist two geodesic paths $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ joining X to Y , with the vertices that precede Y in $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ separated on Σ by Y .*

Using Lemma 1 we will prove:

Lemma 2 *For every genus $g \geq 2$ there exists a pair of curves (X, Y) that fill Σ_g , with X non-separating and Y separating on Σ_g , such that (i) $d(X, Y) = 3$ and (ii) Y is a dead-end of an appropriately chosen geodesic path from X to Y .*

The main results in this paper are:

Theorem 1 *Let X, Y be vertices in $\mathcal{C}(\Sigma)$, with $d(X, Y) = n \geq 3$. Then:*

- (A) *If Y is separating, it may or may not be a dead-end with respect to X .*
- (B) *If Y is non-separating, then any geodesic from X to Y can be extended past Y in arbitrarily many distinct ways, to a geodesic of arbitrary length $> n$.*
- (C) *Conditions (a) and (b) of Lemma 1 are necessary and sufficient for Y to be a dead end with respect to X .*
- (D) *Every dead end in $\mathcal{C}(\Sigma)$ has depth 1.*
- (E) *Examples of dead ends of every length ≥ 3 and double dead-ends of every length ≥ 6 are ubiquitous in $\mathcal{C}(\Sigma)$: For one family of examples, if Y is any non-separating curve on Σ , then every geodesic of length $n \geq 3$ that ends at Y can be extended to a geodesic of length $n + 3$ that terminates in a dead end. Indeed, there are infinitely many such extensions. Moreover, if X is also non-separating, then any geodesic joining X and Y has infinitely many extensions to a geodesic of length $n + 6$ that has a double dead-end.*

Remark 1 Lest the reader misinterpret part (E) of Theorem 1, and think that perhaps there are no double dead ends for distance 3, 4 and 5, we are not suggesting that this is the case. Indeed, we constructed examples of double dead ends with $n = 3$ and 4 by modifying the examples in Lemma 2 so that both X and Y are separating curves. We are not including those examples in this note because there are no general techniques to determine when $d(X, Y) = 4$ or 5, so that new tools were required to explain the examples. We have a second paper in mind, to contribute to the distance problem. It will be joint work with Dan Margalit.

2 Proof of Lemma 2

We must show that there is a basic example for every genus $g > 2$. In our construction we will give curves X, Y, R, L that play the same role as those curves did in the case $g = 2$. In view of Lemma 1, the only work we need to do is to construct the examples.

For a description of Σ_g see Figure 1 for $g = 2$ and 3 and Figure 2 for $g > 3$. In every case Σ_g is represented as a rectangular disc \mathbb{D} with $g - 1$ pairs of interlaced bands attached, so that its boundary is connected and can be capped with a second disc that is covered by the visible one. There are (t_1, t_2, t_3, t_4) pairs of

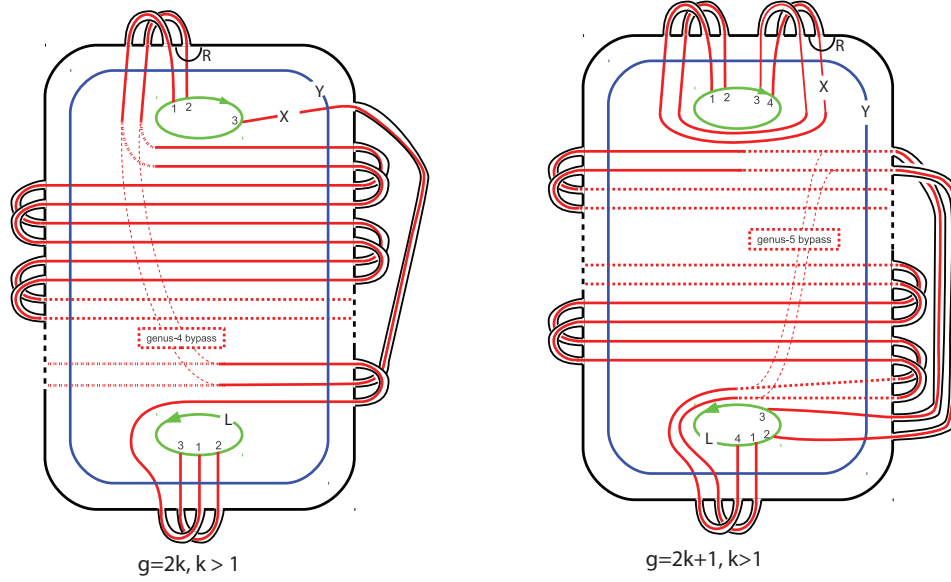


Figure 2: The basic example for genus $g \geq 4$

bands attached on the (top, right, bottom, left) edges of \mathbb{D} , with $\sum_{i=1}^4 t_i = g - 1$. Add 1 to the sum by removing 2 additional discs from the interior of \mathbb{D} and identifying them along their boundaries, using the paired integers. We choose $(t_1, t_2, t_3, t_4) = (1, 0, 0, 0)$ for $g = 2$; $(1, 0, 1, 0)$ for $g = 3$; $(1, 1 + k, 1, k)$ for $g = 4 + 4k$ and $(2, 1 + k, 1, k)$ for $g = 5 + 2k$, where $k = 0, 1, 2, \dots$. Observe that the curve Y splits off a genus 1 subsurface for every $g \geq 2$. The curve R has a uniform description for every g , as in the basic example. The curve L is special for $g = 3$, but uniformly defined for all other g . The curve X is given explicitly for $g = 2$ and 3. For $g = 4$ and 5 it follows the ‘genus 4 bypass’ and ‘genus 5 bypass’. The bypasses are deleted when $k > 0$ and $g > 5$. We leave it to the tireless reader to check that X, Y fill and that X, L and X, R do not fill for every $g \geq 2$. Footnote 3 describes the tool that we used. ■

3 Proof of Theorem 1.

Proof of (A): Consider the basic example on Σ_2 . The vertex Y is a separating vertex that is a dead end for the geodesic $(X.V_L, L, Y)$. On the other hand, the vertex V_L is an interior vertex on the same geodesic that, by construction, is separating because it bounds a subsurface of Σ that is not a disc. Thus it is separating, but it’s not a dead end.

Proof of (B): The key observation that is needed to prove (B) is given in lines

8- and 7- on page 910 of [2]: Assume that Y is non-separating. Choose a vertex $W \in \mathcal{C}(\Sigma)$ such that $d_{\Sigma-Y}(X, W) > M$, where $d_{\Sigma-Y}$ means distance under subsurface projection and where $M = M(\Sigma)$ is the constant given in the BGI Theorem of [2]. Then, as is proved on pages 910-911 of [2], every geodesic joining X to W must pass through Y , i.e. there exists an extension of every geodesic joining X to Y . By making different choices of W we obtain infinitely many such extensions.

Proof of (C): We have already seen, from part (B) of this theorem, that a necessary condition for Y to be a dead end with respect to X is that Y be separating. It remains to prove that if Y is a dead end, then there always exist two geodesic paths $\mathcal{G}^{(1)}, \mathcal{G}^{(2)}$ with the given properties.

Let $\Sigma^{(1)}, \Sigma^{(2)}$ be the 2 components of Σ split along Y . The fact that $n = d(X, Y) \geq 3$ implies that X and Y fill Σ , therefore X intersects both $\Sigma^{(1)}$ and $\Sigma^{(2)}$. Let $\pi : \Sigma \rightarrow \Sigma - \Sigma^{(1)}$ be subsurface projection.

The fact that Σ is closed and of genus ≥ 2 shows that $\mathcal{C}(\Sigma^{(1)}), \mathcal{C}(\Sigma^{(2)})$ both have infinite diameter. This fact, together with part (B), shows that we may choose a non-separating curve $W \subset \Sigma^{(2)}$ with $d_{\Sigma-\Sigma^{(1)}}(X, W) \geq 2n + 1$. It follows from Lemma 2.2 of [4] that any geodesic joining X and W has a vertex $Z^{(1)} \subset \Sigma^{(1)}$. Since $W \subset \Sigma^{(2)}$, it follows that $Z^{(1)} \cap W = Z^{(1)} \cap Y = \emptyset$. But then $d(Z^{(1)}, W) = d(Z^{(1)}, Y) = 1$. This tells us, immediately, that $Z^{(1)}$ must be the vertex that precedes W in any geodesic joining X and W .

We wish to determine $d(X, Z^{(1)})$. Observe that since $W \subset \Sigma^{(2)}$ and $Z^{(1)} \subset \Sigma^{(1)}$, we know that any two of the 3 vertices $Y, W, Z^{(1)}$ are distance 1 apart in $\mathcal{C}(\Sigma)$. By the triangle inequality, $d(X, Y) = n \leq d(X, Z^{(1)}) + 1$. But then, $n - 1 \leq d(X, Z^{(1)})$, which implies that $n \leq d(X, W)$. However, since Y is a dead end, and $Y \cap W = \emptyset$, we also know that $d(X, W) \leq n$, so $d(X, W) = n$. But then $d(X, Z^{(1)}) = n - 1$.

Choose any geodesic path joining X to $Z^{(1)}$. That path extends to the first sought-for geodesic path $\mathcal{G}^{(1)} = (X, \dots, Z^{(1)}, Y)$.

Now observe that if we had assumed that $W \subset \Sigma^{(1)}$ instead of $W \subset \Sigma^{(2)}$, we could have applied the same argument to obtain a second geodesic path $\mathcal{G}^{(2)} = (X, \dots, Z^{(2)}, Y)$. Since $Z^{(1)}$ and $Z^{(2)}$ are separated by Y , we have constructed the required two paths, and proved (C).

Proof of (D): We have just seen, in our proof of (C), that the subpath $(X, \dots, Z^{(1)}) \subset (X, \dots, Z^{(1)}, Y)$ can be extended to the non-separating curve W , and by (B) above arbitrarily far after that. This proves (D).

Proof of (E): We use techniques that we learned from Chapter 2, Sections 5 and 6 of [3]. We are given the geodesic $\mathcal{G} = (V_0, \dots, V_n)$, which ends in the non-separating vertex V_n . Since the initial vertex X in the generalized basic example of Lemma 2 is also non-separating, we may without loss of generality

assume that $X = V_n$. The two geodesic paths \mathcal{L} and \mathcal{R} that are constructed in the basic example give us two paths $\mathcal{G} \circ \mathcal{L}$ and $\mathcal{G} \circ \mathcal{R}$, both extending \mathcal{G} , but the product paths are in general not geodesics. We now consider $(\Sigma \setminus X)$, the surface Σ split along the curve $X = V_n$ and the subsurface projection $\pi : \Sigma \rightarrow (\Sigma \setminus X)$. Choose a pseudo-Anosov map f of $(\Sigma \setminus X)$. For $k > M$ the product paths $\mathcal{G} \circ f^k(\mathcal{L})$ and $\mathcal{G} \circ f^k(\mathcal{R})$ will be geodesics in the curve complex $\mathcal{C}(\Sigma \setminus X)$ which go over to geodesic paths in $\mathcal{C}(\Sigma)$ that extend \mathcal{G} to two geodesics of length $n+3$ that join V_0 to $f^k(Y)$. Since Y is a separating curve it follows that $f^k(Y)$ is too. By Lemma 1, neither geodesic path has an extension beyond $f^k(Y)$.

To build infinitely many such extensions of \mathcal{G} , choose distinct pseudo-Anosov maps f_1, f_2, \dots and powers k_1, k_2, \dots where each $k_i > M$. Then the geodesics $\mathcal{G} \circ f_i^{k_i}(\mathcal{L})$, $i = 1, 2, \dots$ give distinct extensions of \mathcal{G} to geodesics of length $n+3$, where the latter all have dead ends. The construction can be done at each endpoint of \mathcal{G} when both V_0 and V_m are non-separating curves, to give double dead ends. ■

Acknowledgements: We thank Dan Margalit, and also Jason Manning, for patience and help as we struggled to understand recent work on the curve complex; also Yair Minsky for telling us that part (B) of Theorem 1 ought to be true and suggesting how to prove it; also Saul Schleimer for a stimulating discussion about the relationship between our work and his in [4].

The first author gratefully acknowledges support from the Simons Foundation, Award Number 245711.

Joan S. Birman, jb@math.columbia.edu,
William W. Menasco, menasco@buffalo.edu

References

- [1] H.Masur and Y.Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138**, No. 1 (1999), 103-149.
- [2] H.Masur and Y.Minsky, *Geometry of the complex of curves II. Hierarchical structure*, Geom.Funct. Anal. **10**, No. 4 (2000), 902-074.
- [3] S.Schleimer. *Notes on the curve complex* (unpublished lecture notes) <http://homepages.warwick.ac.uk/~masgar/math.html#exposj>.
- [4] S. Schleimer, *The end of the curve complex*, Groups Geom. Dyn. **5** (2011), 169-176.